Computing L-functions of hyperelliptic curves

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Zeta functions and L-functions

Let $X/\mathbb{Q}$ be a nice (smooth, projective, geometrically integral) curve of genus g. For primes p of good reduction (for X) we have a zeta function

$$Z(X_p; s) := \exp\left(\sum_{r\geq 1} \#X_p(\mathbb{F}_{p^r})\frac{T^r}{r}\right) = \frac{L_p(T)}{(1-T)(1-pT)},$$

in which the L-polynomial $L_p \in \mathbb{Z}[T]$ in the numerator satisfies

$$L_p(T) = T^{2g}\chi_p(1/T) = 1 - a_pT + \cdots + p^gT^{2g};$$

here $\chi_p(T)$ is the charpoly of the Frobenius endomorphism of $\mathrm{Jac}(X_p)$ (this implies $\#\mathrm{Jac}(X_p) = L_p(1)$, for example). The L-function of X is

$$L(X,s) = L(\mathrm{Jac}(X), s) := \sum_{n\geq 1} a_n n^{-s} := \prod_p L_p(p^{-s})^{-1},$$

where the Dirichlet coefficients $a_n \in \mathbb{Z}$ are determined by the $L_p(T)$. In particular, $a_p = p + 1 - \#X_p(\mathbb{F}_p)$ is the trace of Frobenius.

The Selberg class with polynomial Euler factors

The Selberg class S^{poly} consists of Dirichlet series $L(s) = \sum_{n\geq 1} a_n n^{-s}$:

1. $L(s)$ has an analytic continuation that is holomorphic at $s \neq 1$;
2. For some $\gamma(s) = Q^s \prod_{i=1}^{r} \Gamma(\lambda_i s + \mu_i)$ and ε, the completed L-function $\Lambda(s) := \gamma(s)L(s)$ satisfies the functional equation

$$\Lambda(s) = \varepsilon\overline{\Lambda(1-\bar{s})},$$

   where $Q > 0$, $\lambda_i > 0$, $\mathrm{Re}(\mu_i) \geq 0$, $|\varepsilon| = 1$. Define $\deg L := 2\sum_i^r \lambda_i$.
3. $a_1 = 1$ and $a_n = O(n^{\epsilon})$ for all $\epsilon > 0$ (Ramanujan conjecture).
4. $L(s)$ has an Euler product $L(s) = \prod_p L_p(p^{-s})^{-1}$ in which each local factor $L_p \in \mathbb{Z}[T]$ has degree at most $\deg L$.

The Dirichlet series $L_{\text{an}}(s, X) := L(X, s + \frac{1}{2})$ satisfies (3) and (4), and conjecturally lies in S^{poly}; for $g = 1$ this is known (via modularity).

Strong multiplicity one

Theorem (Kaczorowski-Perelli 2001)

If $A(s) = \sum_{n\geq 1} a_n n^{-s}$ *and* $B(s) = \sum_{n\geq 1} b_n n^{-s}$ *lie in* S^{poly} *and* $a_p = b_p$ *for all but finitely many primes* p*, then* $A(s) = B(s)$*.*

Corollary

If $L_{\mathrm{an}}(s,X)$ *lies in* S^{poly} *then it is determined by (any choice of) all but finitely many coefficients* a_p*. In particular, all of the local factors are completely determined by the Frobenius traces* a_p *at good primes.*

Henceforth we assume that $L_{\mathrm{an}}(s,X) \in S^{\mathrm{poly}}$.

Let $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^s\Gamma(s)$, and define $\Lambda(X,s) := \Gamma_{\mathbb{C}}(s)^g L(X,s)$. Then

$$\Lambda(X,s) = \varepsilon N^{1-s}\Lambda(X,2-s).$$

where the analytic root number $\varepsilon = \pm 1$ and analytic conductor $N \in \mathbb{Z}_{\geq 1}$ are also determined by the Frobenius traces a_p at good primes.

Effective strong multiplicity one

Fix a finite set of primes $\mathcal{S}$ (e.g. bad primes) and an integer M that we know is a multiple of the conductor N (e.g. $M = \Delta(X)$).

There is a finite set of possibilities for $\varepsilon = \pm 1$, $N|M$, and the Euler factors $L_p \in \mathbb{Z}[T]$ for $p \in \mathcal{S}$ (the coefficients of $L_p(T)$ are bounded).

Suppose we know the a_n for all $n \leq c_1\sqrt{M}$ with $p \nmid n$ for $p \in \mathcal{S}$. For a suitably large c_1, exactly one choice of ε, N, and $L_p(T)$ for $p \in \mathcal{S}$ will make it possible for $L(X, s)$ to satisfy its functional equation.[1]

One can explicitly determine a set of $O(N^\epsilon)$ candidate values of c_1, one of which is guaranteed to work; in practice the first one usually works.

This gives an effective algorithm to compute ε, N, and $L_p(T)$ for $p \in \mathcal{S}$, provided we can compute $L_p(T)$ at good $p \leq B$, where $B = O(\sqrt{N})$.

[1] Subject to our assumption that $L_{\text{an}} \in \mathcal{S}^{\text{poly}}$. But if the algorithm fails we have an explicit counterexample to the conjectured Langlands correspondence.

Algorithms to compute zeta functions

Given $X/\mathbb{Q}$ of genus g, we want to compute $L_p(T)$ for all good $p \le B$.

	complexity per prime (ignoring $(\log\log p)^{O(1)}$ factors)		
algorithm	$g = 1$	$g = 2$	$g = 3$
point enumeration	$p \log p$	$p^2 \log p$	$p^3 (\log p)^2$
group computation	$p^{1/4} \log p$	$p^{3/4} \log p$	$p (\log p)^2$
p-adic cohomology	$p^{1/2} (\log p)^2$	$p^{1/2} (\log p)^2$	$p^{1/2} (\log p)^2$
CRT (Schoof-Pila)	$(\log p)^5$	$(\log p)^8$	$(\log p)^{12?}$
average poly-time	$(\log p)^4$	$(\log p)^4$	$(\log p)^4$

For $L(X,s) = \sum a_n n^{-s}$, we only need a_{p^2} for $p^2 \le B$, and a_{p^3} for $p^3 \le B$.
For $1 < r \le g$ we can easily compute a_{p^r} for $p^r \le B$ in time $O(B \log B)$.

Bottom line: it all comes down to computing Frobenius traces.

Warmup: average polynomial-time in genus 1

Let $X : y^2 = f(x)$ with $\deg f = 3, 4$ and $f(0) \neq 0$, and let f_k^n denote the coefficient of x^k in f^n. For each good prime p we have

$$\begin{aligned}
\#X_p(\mathbb{F}_p) &= \sum_{a,b\in\mathbb{F}_p} \left[b^2 = f(a)\right] + N_\infty \\
&\equiv \sum_{a,b\in\mathbb{F}_p} \left(1 - (b^2 - f(a))^{p-1}\right) + N_\infty \\
&\equiv -\sum_{a,b\in\mathbb{F}_p} \sum_r \binom{p-1}{r} b^{2r}(-f(a))^{p-1-r} + N_\infty \\
&\equiv \binom{p-1}{\frac{p-1}{2}} (-1)^{\frac{p-1}{2}} \sum_{a\in\mathbb{F}_p} f(a)^{\frac{p-1}{2}} + N_\infty \\
&\equiv \left(-f_{2p-2}^{(p-1)/2} - f_{p-1}^{(p-1)/2}\right) + \left(1 + (f_4)^{(p-1)/2)}\right) \\
&\equiv 1 - f_{p-1}^{(p-1)/2} \equiv 1 - a_p
\end{aligned}$$

Thus $a_p \equiv f_{p-1}^{(p-1)/2}$. This determines $a_p \in \mathbb{Z}$ for $p \geq 17$, since $|a_p| \leq 2\sqrt{p}$.

Warmup: average polynomial-time in genus 1

We want to compute $f_{p-1}^{(p-1)/2}$ modulo p for many primes p.

The relations $f^{n+1} = f \cdot f^n$ and $(f^{n+1})' = (n+1)f' \cdot f^n$ yield the identity

$$kf_0 f_k^n = \sum_{1 \le i \le d} (i(n+1) - k) f_i f_{k-i}^n,$$

valid for all $k, n \ge 0$. For $d = 3$ (and similarly for $d = 4$), we define

$$v_k^n := [f_{k-2}^n, f_{k-1}^n, f_k^n], \qquad M_k^n := \begin{bmatrix} 0 & 0 & (3n+3-k)f_3 \\ kf_0 & 0 & (2n+2-k)f_2 \\ 0 & kf_0 & (n+1-k)f_1 \end{bmatrix}.$$

For all positive integers k and n we then have

$$v_k^n = \frac{1}{kf_0} v_{k-1}^n M_k^n = \frac{1}{(f_0)^k k!} v_0^n M_1^n \cdots M_k^n.$$

Warmup: average polynomial-time in genus 1

We want to compute $a_p \equiv f_{2n}^n \bmod p$ with $n := (p-1)/2$.
This is the last entry of the vector

$$v_{2n}^n = \frac{1}{f_0^{2n}(2n!)} v_0^n M_1^n \cdots M_{2n}^n = -v_0^n M_1^n \cdots M_{2n}^n$$

reduced modulo $p = 2n+1$.

Observe that $2(n+1) \equiv 1 \bmod p$, so $2M_k^n \equiv M_k \bmod p$, where

$$M_k := \begin{bmatrix} 0 & 0 & (3-2k)f_3 \\ kf_0 & 0 & (2-2k)f_2 \\ 0 & kf_0 & (1-2k)f_1 \end{bmatrix} \in \mathbb{Z}^{3\times 3}$$

is independent of $\boldsymbol{p}$. For each odd prime $p = 2n+1$ we have

$$v_{2n}^n \equiv -\left(\frac{f_0}{p}\right) V_0 M_1 \cdots M_{2n-1} \bmod p \qquad \text{(where } V_0 = [0,0,1]\text{)}.$$

Accumulating remainder tree

Given matrices $M_0, \ldots, M_{n-1}$ and moduli $m_1, \ldots, m_n$, to compute

$$\begin{aligned} M_0 &\bmod m_1 \\ M_0 M_1 &\bmod m_2 \\ M_0 M_1 M_2 &\bmod m_3 \\ M_0 M_1 M_2 M_3 &\bmod m_4 \\ &\cdots \\ M_0 M_1 \cdots M_{n-2} M_{n-1} &\bmod m_n \end{aligned}$$

multiply adjacent pairs and recursively compute

$$\begin{aligned} (M_0 M_1) &\bmod m_2 m_3 \\ (M_0 M_1)(M_2 M_3) &\bmod m_4 m_5 \\ &\cdots \\ (M_0 M_1) \cdots (M_{n-2} M_{n-1}) &\bmod m_n \end{aligned}$$

and adjust the results as required (for better results, use a forest).

Complexity analysis

Assume $\log|f_i| = O(\log B)$. The recursion has depth $O(\log B)$, and in each recursive step we multiply and reduce a bunch of 3×3 matrices with integer entries whose total bitsize is $O(B \log B)$.

We can do all the multiplications/reductions at any given level of the recursion using $O(\mathsf{M}(B \log B)) = B(\log B)^{2+o(1)}$ bit operations.

Total complexity is $B(\log B)^{3+o(1)}$, or $(\log p)^{4+o(1)}$ per prime $p \leq B$.

For a single prime p we do not have a polynomial-time algorithm, but we can give an $O(p^{1/2}(\log p)^{1+o(1)})$ algorithm using the same matrices.

This is a silly way to compute a single a_p in genus 1, but its generalization to genus 2 is competitive, and in genus 3 it yields the fastest practical method known (within the feasible range of p).

Efficiently handling a single prime

Simply computing $V_0 M_1 \cdots M_{p-1}$ modulo p is surprisingly quick (faster than semi-naïve point-counting); it takes $p(\log p)^{1+o(1)}$ time.
For small p (into the thousands), this is the the fastest approach.
But we can do better.

Viewing $M_k \bmod p$ as $M \in \mathbb{F}_p[k]^{3\times 3}$, we compute

$$A(k) := M(k)M(k+1)\cdots M(k+r-1) \in \mathbb{F}_p[k]^{3\times 3}$$

with $r \approx \sqrt{p}$, pick $s \approx \sqrt{p}$ so $rs < p$ and evaluate $A(k)$ at $s = \lfloor p/r \rfloor$ points to get

$$M_1 M_2 \cdots M_{p-1} \equiv_p A(1)A(r+1)A(2r+1)\cdots A((s-1)r+1)M_{sr+1}\cdots M_{p-1}.$$

Using standard product tree and multipoint evaluation techniques this takes $O(\mathsf{M}(p^{1/2})\log p) = p^{1/2}(\log p)^{2+o(1)}$ time.

Bostan-Gaudry-Schost: $p^{1/2}(\log p)^{1+o(1)}$ time.

The Hasse-Witt matrix of a hyperelliptic curve

Let $X_p/\mathbb{F}_p$ be a hyperelliptic curve $y^2 = f(x)$ of genus g (with $p \neq 2$).
As in the warmup, let f_k^n denote the coefficient of x^k in f^n.

The Hasse–Witt matrix of X_p is $W_p := [f_{pi-j}^n]_{ij} \in \mathbb{F}_p^{g\times g}$ with $n = (p-1)/2$.
By the same argument used in our warmup, have

$$a_p \equiv \sum_{i=1}^{g} f_{ip-i}^n = \operatorname{tr} W_p,$$

and for $p > 16g^2$ this uniquely determines $a_p \in \mathbb{Z}$.

In fact, as proved by Manin, we have

$$L_p(T) \equiv \det(I - TW_p) \bmod p.$$

One can define the Hasse-Witt matrix for curve over $\mathbb{F}_p$: it is the matrix of the p-power Frobenius acting on $H^1(X_p, \mathcal{O}_{X_p})$, and via Serre duality, the matrix of the Cartier-Manin operator acting on Ω_{X_p}.

Hyperelliptic average polynomial-time

As in our warmup, assume $f(0) \neq 0$ and define $v_k^n := [f_{k-d+1}^n, \ldots, f_k^n]$. The last g entries of v_{2n}^n form the first row of W_p, and we have

$$v_{2n}^n = -\left(\frac{f_0}{p}\right) V_0 M_1 \cdots M_{p-1} \bmod p \qquad \text{(where } V_0 = [0, \ldots, 0, 1]).$$

Compute the first row of W_p for good $p \leq B$ in $O(g^2 B (\log B)^{3+o(1)})$ time.

To get the remaining rows, consider the isomorphic curve $y^2 = f(x+a)$ whose Hasse-Witt matrix $W_p(a) = T(a) W_p T(-a)$ is conjugate to W_p via

$$T(a) := \left[\binom{j-1}{i-1} a^{j-1}\right]_{ij} \in \mathbb{F}_p^{g \times g}.$$

Given the first row of $W_p(a)$ for g distinct values of a we can compute all the rows of W_p. Total complexity is $O(g^3 B (\log B)^{3+o(1)})$, with an average complexity of $O(g^3 p^{4+o(1)})$, which is polynomial in both g and $\log p$.

Analytic rank data for elliptic/hyperelliptic curves

Genus 1 curves with $|\Delta_{\min}| \le 10^5: 17,247$ (exact count)

Genus 2 curves with $|\Delta_{\min}| \le 10^6: 66,158$ (lower bound)

Genus 3 curves with $|\Delta_{\min}| \le 10^7: 67,879$ (lower bound)

	genus 1		genus 2		genus 3	
rank	count	percent	count	percent	count	percent
0	6408	37,15	12131	18.34	7770	11.45
1	8586	49.78	30579	46.22	30840	45.47
2	2182	12.65	20561	31.08	25486	30.11
3	71	0.41	2877	4.35	3723	5.49
4	0	0.00	10	0.02	8	0.01

Genus 2 data includes all $y^2 + h(x)y = f(x)$ with $\|h\|_\infty \le 1, \|f\|_\infty \le 90$.
Genus 3 data includes all $y^2 + h(x)y = f(x)$ with $\|h\|_\infty \le 1, \|f\|_\infty \le 31$.

Genus 3 curves

The canonical embedding of a genus 3 curve into $\mathbb{P}^2$ is either

1. a degree-2 cover of a smooth conic (hyperelliptic case);
2. a smooth plane quartic (generic case).

Average polynomial-time implementations now available for all cases:

- rational hyperelliptic model $y^2 = f(x)$ [Harvey-S 2014].
- degree-2 cover of a smooth conic [Harvey-Massierer-S 2016].
- smooth plane quartic [Harvey-S 2017].

Essentially all prior work in genus 3 uses p-adic cohomology:

[Kedlaya 2001], [Gaudry-Gürel 2003], [Lauder 2004], [Kedlaya 2006], [Castryck-Denef-Vercauteren 2006], [Abbott-Kedlaya-Roe 2006], [Harvey 2007], [Hubrechts 2007], [Harvey 2010], [Hubrechts 2011], [Harrison 2012], [Tuitman-Pancrantz 2013], [Tuitman 2015], [Costa 2015], [Tuitman-Castryck 2016], [Shieh 2016]

The Hasse-Witt matrix of a smooth plane quartic

Let $X_p/\mathbb{F}_p$ be a smooth plane quartic defined by $f(x,y,z)=0$.
For $n \geq 0$ let $f^n_{i,j,k}$ denote the coefficient of $x^i y^j z^k$ in f^n.

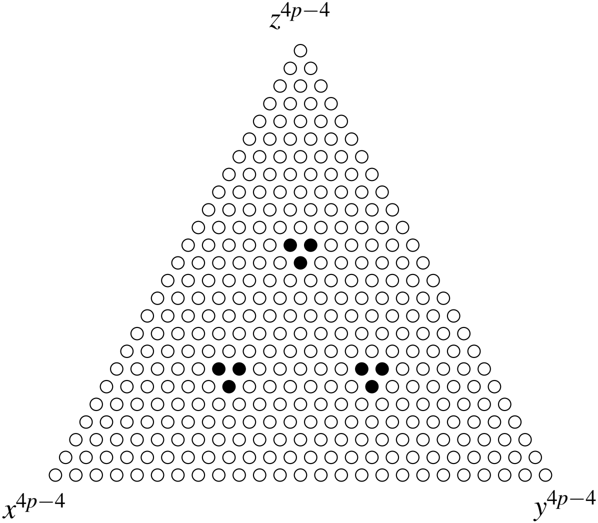
The Hasse–Witt matrix of X_p is the 3×3 matrix

$$W_p := \begin{bmatrix} f^{p-1}_{p-1,p-1,2p-2} & f^{p-1}_{2p-1,p-1,p-2} & f^{p-1}_{p-1,2p-1,p-2} \\ f^{p-1}_{p-2,p-1,2p-1} & f^{p-1}_{2p-2,p-1,p-1} & f^{p-1}_{p-2,2p-1,p-1} \\ f^{p-1}_{p-1,p-2,2p-1} & f^{p-1}_{2p-1,p-2,p-1} & f^{p-1}_{p-1,2p-2,p-1} \end{bmatrix}.$$

This case of smooth plane curves of degree $d > 4$ is similar.

More generally, given a singular plane model for any nice curve (equivalently, a defining polynomial for its function field) one can use the methods of Stohr-Voloch to explicitly determine W_p.

Target coefficients of f^{p-1} for $p = 7$:



Coefficient relations

Let $\partial_x = x\frac{\partial}{\partial x}$ (degree-preserving). The relations

$$f^{p-1} = f \cdot f^{p-2} \qquad \text{and} \qquad \partial_x f^{p-1} = -(\partial_x f) f^{p-2}$$

yield the relation

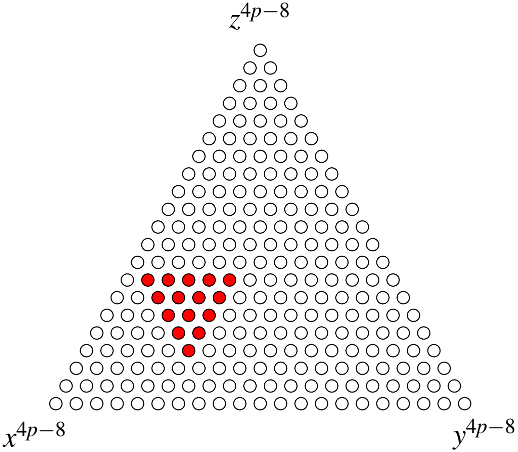
$$\sum_{i'+j'+k'=4} (i+i') f_{i',j',k'} f^{p-2}_{i-i',j-j',k-k'} = 0.$$

among nearby coefficients of f^{p-2} (a triangle of side length 5).

Replacing ∂_x by ∂_y yields a similar relation (replace $i+i'$ with $j+j'$).

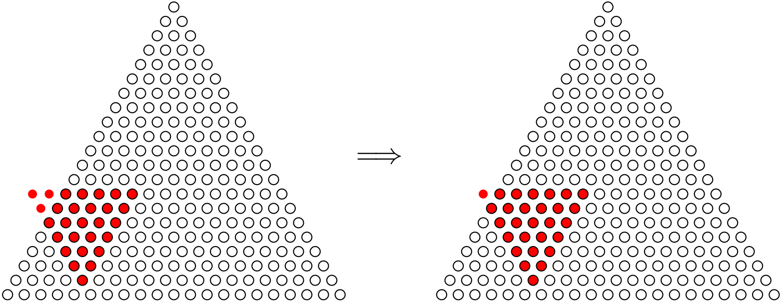
Coefficient triangle

For $p = 7$ with $i = 12, j = 5, k = 7$ the related coefficients of f^{p-2} are:



Moving the triangle

Now consider a bigger triangle with side length 7.
Our relations allow us to move the triangle around:



An initial “triangle” at the edge can be efficiently computed using coefficients of $f(x, 0, z)^{p-2}$.

Computing one Hasse-Witt matrix

Nondegeneracy: we need $f(1,0,0), f(0,1,0), f(0,0,1)$ nonzero and $f(0,y,z), f(x,0,z), f(x,y,0)$ squarefree (easily achieved for large p).

The basic strategy to compute W_p is as follows:

- There is a 28×28 matrix M_j that shifts our 7-triangle from y-coordinate j to $j+1$; its coefficients depend on j and f.
  In fact a 16×16 matrix M_i suffices (use smoothness of C).
- Applying the product $M_0 \cdots M_{p-2}$ to an initial triangle on the edge and applying a final adjustment to shift from f^{p-2} to f^{p-1} gets us one column of the Hasse-Witt matrix W_p.
- By applying the same product (or its inverse) to different initial triangles we can compute all three columns of W_p.

We have thus reduced the problem to computing $M_1 \cdots M_{p-2} \bmod p$, which we already know how to do, either in $p^{1/2}(\log p)^{1+o(1)}$ time, or in average polynomial time $(\log p)^{4+o(1)}$.

Cumulative timings for genus 3 curves

Time to compute $L_p(T) \bmod p$ for all good $p \leq B$.

B	spq-Costa-AKR	spq-HS	ghyp-MHS	hyp-HS	hyp-Harvey
2^{12}	18	1.4	0.3	0.1	1.3
2^{13}	49	2.4	0.7	0.2	2.6
2^{14}	142	4.6	1.7	0.5	5.4
2^{15}	475	9.4	4.6	1.0	12
2^{16}	1,670	21	11	2.1	29
2^{17}	5,880	47	27	5.3	74
2^{18}	22,300	112	62	14	192
2^{19}	78,100	241	153	37	532
2^{20}	297,000	551	370	97	1,480
2^{21}	1,130,000	1,240	891	244	4,170
2^{22}	4,280,000	2,980	2,190	617	12,200
2^{23}	16,800,000	6,330	5,110	1,500	36,800
2^{24}	66,800,000	14,200	11,750	3,520	113,000
2^{25}	244,000,000	31,900	28,200	8,220	395,000
2^{26}	972,000,000	83,300	62,700	19,700	1,060,000

(Intel Xeon E7-8867v3 3.3 GHz CPU seconds).